

# JACOBIAN FIBRATIONS ON THE SINGULAR $K3$ SURFACE OF DISCRIMINANT 3

KAZUKI UTSUMI

ABSTRACT. In this paper we give the Weierstrass equations and the generators of Mordell-Weil groups for Jacobian fibrations on the singular  $K3$  surface of discriminant 3.

## 1. INTRODUCTION

A  $K3$  surface defined over the complex number field whose Picard number equals to maximum possible number 20 is called a *singular  $K3$  surface*. Shioda and Inose [10] showed that the map a singular  $K3$  surface  $X$  corresponds to its transcendental lattice  $T_X$  is a bijective correspondence from the set of singular  $K3$  surfaces onto the set of equivalence classes of positive-definite even integral lattice of rank two with respect to  $SL_2(\mathbb{Z})$ . The discriminant of a singular  $K3$  surface  $X$  is the determinant of the Gram matrix of the transcendental lattice  $T_X$ .

In this paper we study Jacobian fibrations, i.e., elliptic fibrations with a section, on the singular  $K3$  surface  $X_3$  of discriminant 3, which corresponds to the lattice defined by  $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$  and is uniquely determined up to isomorphism. Jacobian fibrations on  $X_3$  were classified by Nishiyama [8]. He classified all configurations of singular fibers of Jacobian fibrations on  $X_3$  into 6 classes and determined their Mordell-Weil groups. Then, we give for each fibration a Weierstrass model. More precisely, we state our main theorem.

**Theorem 1.** *Let  $X_3$  be the singular  $K3$  surface of discriminant 3. For each Jacobian fibration in Nishiyama's list [8, Table 1.1], an elliptic parameter  $u_i$ , a Weierstrass equation and the generators of the Mordell-Weil group are given by Table 1.*

We explain about Table 1. The first column shows the name of each Jacobian fibrations following Nishiyama's notation. The second column shows the configuration of singular fibers. Here, for example, by  $2II^* + IV$  means that the surface has two singular fibers of type  $II^*$  and a singular fiber of type  $IV$  (Kodaira's notation [4]). The third column shows the Mordell-Weil group (MWG) of the fibration. The fourth column shows an elliptic parameter  $u_i$  of the fibration under the singular affine model (2.6) of  $X_3$ . The index  $i$  is the name of the fibration. The last column shows a Weierstrass equation and rational points corresponding to Mordell-Weil generator of the fibration, where  $O$  is the rational point corresponding to the zero of MWG.

Recently, Braun, Kimura and Watari [2] showed that Nishiyama's list also gives the classification of Jacobian fibrations on  $X_3$  modulo isomorphism. Thus, our and their results answer completely a question of Kuwata and Shioda [7].

No.	sing. fibs	MWG	$u_i$	equation and rational points
1	$2 \text{ II}^* + \text{IV}$	0	$\frac{2(y_2+1)}{(y_1-1)^2}$	$Y^2 = X^3 + u_1^5(u_1 - 1)^2$ $O$
2	$\text{I}_{12}^* + \text{I}_3 + 3 \text{ I}_1$	$\mathbb{Z}/2\mathbb{Z}$	$\frac{2t^2}{(y_2+1)(y_1^2+2y_1+2y_2-1)}$	$Y^2 = X^3 - 2u_2(u_2^3 - 2)X^2 + u_2^8 X$ $O, (0, 0)$
3	$\text{III}^* + \text{I}_6^* + 3 \text{ I}_1$	$\langle \frac{3}{2} \rangle \oplus \mathbb{Z}/2\mathbb{Z}$	$\frac{t}{y_1^2-1}$	$Y^2 = X^3 + 4u_3^3 X^2 - 4u_3^3 X$ 2-tor.: $O, (0, 0)$ free gen. : $(1, -1)$
4	$\text{I}_{18} + 6 \text{ I}_1$	$\langle \frac{3}{2} \rangle \oplus \mathbb{Z}/3\mathbb{Z}$	$\frac{t}{y_1+y_2}$	$Y^2 = X^3 + (X - u_4^6)^2$ 3-tor. : $O, (0, \pm u_4^6)$ free gen. : $(2u_4^3, 2u_4^3 + u_4^6)$
5	$3 \text{ IV}^*$	$\mathbb{Z}/3\mathbb{Z}$	$y_1$	$Y^2 = X^3 + (u_5^2 - 1)^4$ $O, (0, \pm(u_5^2 - 1)^2)$
6	$\text{I}_3^* + \text{I}_{12} + 3 \text{ I}_1$	$\mathbb{Z}/4\mathbb{Z}$	$t$	$Y^2 = X^3 - 2(u_6^3 - 2)X^2 + u_6^6 X$ $O, (0, 0), (u_6^3, \pm 2u_6^6)$

TABLE 1. Classification of Jacobian fibrations on  $X_3$ 

## 2. NOTATION

The singular  $K3$  surface  $X_3$  is known as a *generalized Kummer surface* constructed by the following. Let  $C_\omega$  be the complex elliptic curve with the fundamental periods 1 and  $\omega = e^{2\pi\sqrt{-1}/3}$ . Let  $\sigma$  be an automorphism of  $C_\omega \times C_\omega$  defined by  $\sigma(z_1, z_2) \mapsto (\omega z_1, \omega^2 z_2)$ . Then the minimal resolution of the quotient  $C_\omega \times C_\omega / \langle \sigma \rangle$  is isomorphic to the singular  $K3$  surface  $X_3$  (see [10, Lemma 5.1]). The automorphism  $\sigma$  has the 9 fixed points  $(v_i, v_j)$  ( $1 \leq i, j \leq 3$ ), where  $\{v_i\}$  are the fixed points of the automorphism  $\sigma_1$  of  $C_\omega$  defined by  $\sigma_1(z) = \omega z$ . These 9 points  $(v_i, v_j)$  correspond to the singular points  $p_{ij}$  of the quotient  $C_\omega \times C_\omega / \langle \sigma \rangle$ . The minimal resolution  $X_3$  of  $C_\omega \times C_\omega / \langle \sigma \rangle$  is obtained by replacing each  $p_{ij}$  by 2 non-singular rational curves  $E_{i,j}$  and  $E'_{i,j}$  with  $E_{i,j} \cdot E'_{i,j} = 1$ . Moreover,  $X_3$  contains 6 non-singular rational curves, i.e. the image  $F_i$  (or  $G_j$ ) of  $\{v_i\} \times C_\omega$  (or  $C_\omega \times \{v_j\}$ ) in  $X_3$ . We have the following intersection numbers.

$$(2.1) \quad \begin{aligned} F_i^2 = G_i^2 = E_{i,j}^2 = E'_{i,j}{}^2 = -2, \quad F_i \cdot E_{j,k} = G_i \cdot E'_{j,k} = F_i \cdot G_j = 0, \\ E_{i,j} \cdot E'_{k,l} = \delta_{i,k} \cdot \delta_{j,l}, \quad F_i \cdot E'_{j,k} = G_i \cdot E_{k,j} = \delta_{i,j}. \end{aligned}$$

These 24 curves on  $X_3$  form the configuration of Figure 1.

It is well known that the elliptic curve  $C_\omega$  has the following Weierstrass form

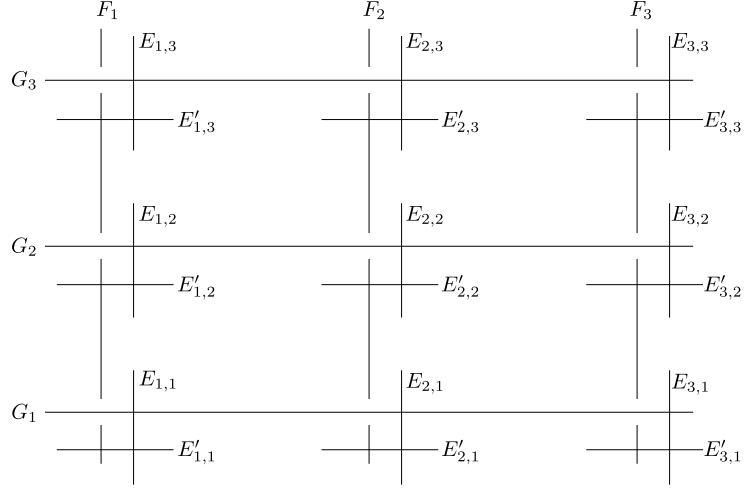
$$(2.2) \quad C_\omega : y^2 = x^3 + 1.$$

We denote each factor of  $C_\omega \times C_\omega$  by

$$(2.3) \quad C_\omega^1 : y_1^2 = x_1^3 + 1, \quad C_\omega^2 : y_2^2 = x_2^3 + 1.$$

Then the automorphism  $\sigma$  is written by

$$(2.4) \quad \begin{aligned} \sigma : C_\omega^1 \times C_\omega^2 &\rightarrow C_\omega^1 \times C_\omega^2 \\ (x_1, y_1, x_2, y_2) &\mapsto (\omega x_1, y_1, \omega^2 x_2, y_2). \end{aligned}$$

FIGURE 1.  $(-2)$ -curves

The function field  $\mathbb{C}(X_3)$  is equal to the invariant subfield of the function field  $\mathbb{C}(C_\omega^1 \times C_\omega^2) = \mathbb{C}(x_1, x_2, y_1, y_2)$  under the automorphism  $\sigma$ . Then we have

$$(2.5) \quad \mathbb{C}(X_3) = \mathbb{C}(y_1, y_2, t), \quad t = x_1 x_2,$$

where  $y_1, y_2$ , and  $t$  are naturally regarded as functions on  $X_3$  with the relation

$$(2.6) \quad t^3 = (y_1^2 - 1)(y_2^2 - 1).$$

This gives a singular affine model of  $X_3$ . We start from the equation to obtain a Weierstrass form for each Jacobian fibration on  $X_3$ . Under the above notation, we

see that the divisor of typical functions are as follows.

(2.7)

$$(y_1 - 1) = 3F_2 + 2(E'_{2,1} + E'_{2,2} + E'_{2,3}) + E_{2,1} + E_{2,2} + E_{2,3} \\ - (3F_1 + 2(E'_{1,1} + E'_{1,2} + E'_{1,3}) + E_{1,1} + E_{1,2} + E_{1,3})$$

$$(y_1 + 1) = 3F_3 + 2(E'_{3,1} + E'_{3,2} + E'_{3,3}) + E_{3,1} + E_{3,2} + E_{3,3} \\ - (3F_1 + 2(E'_{1,1} + E'_{1,2} + E'_{1,3}) + E_{1,1} + E_{1,2} + E_{1,3})$$

$$(y_2 - 1) = 3G_2 + 2(E_{1,2} + E_{2,2} + E_{3,2}) + E'_{1,2} + E'_{2,2} + E'_{3,2} \\ - (3G_1 + 2(E_{1,1} + E_{2,1} + E_{3,1}) + E'_{1,1} + E'_{2,1} + E'_{3,1})$$

$$(y_2 + 1) = 3G_3 + 2(E_{1,3} + E_{2,3} + E_{3,3}) + E'_{1,3} + E'_{2,3} + E'_{3,3} \\ - (3G_1 + 2(E_{1,1} + E_{2,1} + E_{3,1}) + E'_{1,1} + E'_{2,1} + E'_{3,1})$$

$$(t) = F_2 + E'_{2,3} + E_{2,3} + G_3 + E_{3,3} + E'_{3,3} + F_3 + E'_{3,2} + E_{3,2} + G_2 + E_{2,2} + E'_{2,2} \\ - (E_{2,1} + E_{3,1} + 2(G_1 + E_{1,1} + E'_{1,1} + F_1) + E'_{1,2} + E'_{1,3}).$$

## 3. FIBRATION 1

An elliptic parameter for Fibration 1 is given by

$$(3.1) \quad u_1 = \frac{2(y_1 + 1)}{(y_1 - 1)^2}.$$

The divisor of  $u_1$  is given by

$$(3.2) \quad \begin{aligned} (u_1) = & E'_{3,3} + 2E_{3,3} + 3G_3 + 4E_{1,3} + 5E'_{1,3} + 6F_1 + 3E'_{1,1} + 4E'_{1,2} + 2E_{1,2} \\ & - (E'_{3,1} + 2E_{3,1} + 3G_1 + 4E_{2,1} + 5E'_{2,1} + 6F_2 + 3E'_{2,3} + 4E'_{2,2} + 2E_{2,2}). \end{aligned}$$

The zero divisor  $(u_1)_0$  (the bold lines in Figure 2) and the polar divisor  $(u_1)_\infty$  (the thin lines in Figure 2) are the singular fibers both of type  $\text{II}^*$ .

Eliminating the variable  $y_2$  from (2.6) and (3.1), we obtain the following equation

$$(3.3) \quad 4t^3 = u_1(y_1 + 1)(y_1 - 1)^3(u_1 y_1^2 - 2u_1 y_1 + u_1 - 4),$$

which defines a plane curve over  $\mathbb{C}(u_1)$  with a singularity at  $(t, y_1) = (0, 1)$ . Blowing up by  $t = v(y_1 - 1)$ , we have the following equation

$$(3.4) \quad 4v^3 = u_1(y_1 + 1)(u_1 y_1^2 - 2u_1 y_1 + u_1 - 4),$$

which defines a nonsingular plane cubic curve over  $\mathbb{C}(u_1)$  with a rational point  $(v, y_1) = (0, -1)$ . Then we can convert it into a Weierstrass form (see [1] or [3]). Since the rational point  $(v, y_1) = (0, -1)$  corresponds to the divisor  $F_3$  (the dotted line in Figure 2), choosing it as the zero section of the group structure, we obtain the Weierstrass equation for Fibration 1

$$(3.5) \quad Y^2 = X^3 + u_1^5(u_1 - 1)^2,$$

where the change of variables is given by

$$(3.6) \quad X = \frac{\sqrt[3]{4}(u_1 - 1)u_1 t}{(y_1^2 - 1)}, \quad Y = -\frac{u_1^2(u_1 - 1)(u_1 y_1 - u_1 + 2)}{y_1 + 1}.$$

Besides the two singular fibers of type  $\text{II}^*$  at  $u_1 = 0$  and  $\infty$ , there is one singular fiber of type IV at  $u_1 = 1$ . It is the divisor  $E_{3,2} + E'_{3,2} + Q_1$  (the long dashed dotted lines in Figure 2), where  $Q_1$  is a  $(-2)$ -curve on  $X_3$  arising from a curve on  $\mathbb{P}^1 \times \mathbb{P}^1$  below.

Let  $p_j : C_\omega^j \rightarrow \mathbb{P}^1$  ( $j = 1, 2$ ) be the projection given by

$$(3.7) \quad \begin{aligned} p_j : C_\omega^j & \rightarrow \mathbb{P}^1 \\ (x_j : y_j : z_j) & \mapsto \begin{cases} (y_j : z_j) & \text{if } z_j \neq 0 \\ (1 : 0) & \text{if } z_j = 0. \end{cases} \end{aligned}$$

Then the map  $p_1 \times p_2 : C_\omega^1 \times C_\omega^2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  factors through  $\bar{\pi} : C_\omega^1 \times C_\omega^2 / \sigma \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\pi$  be the morphism of degree three from  $X_3$  to  $\mathbb{P}^1 \times \mathbb{P}^1$  that makes the following diagram commutative:

$$\begin{array}{ccccc} & & X_3 & & \\ & & \downarrow & \searrow \pi & \\ C_\omega^1 \times C_\omega^2 & \longrightarrow & C_\omega^1 \times C_\omega^2 / \sigma & \xrightarrow{\bar{\pi}} & \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

It is easy to verify that the equation  $u_1 = 1$  means

$$(3.8) \quad y_1^2 - 2y_1 - 2y_2 - 1 = 0$$

from (3.1). This equation defines a curve on  $\mathbb{P}^1 \times \mathbb{P}^1$ . Then it lifts to the  $(-2)$ -curve  $Q_1$  on  $X_3$  via the map  $\pi$ .

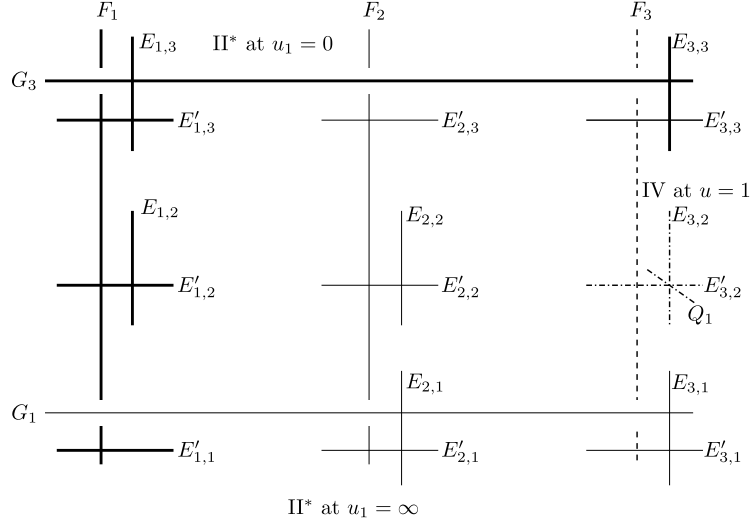


FIGURE 2. Fibration 1

#### 4. FIBRATION 3

An elliptic parameter for Fibration 3 is given by

$$(4.1) \quad u_3 = \frac{t}{y_1^2 - 1}.$$

The divisor of  $u_3$  is given by

$$(4.2) \quad (u_3) = G_2 + 2E_{1,2} + 3E'_{1,2} + 4F_1 + 3E'_{1,1} + 2E_{1,3} + G_3 + 3E'_{1,2} - (E'_{2,2} + E'_{2,3} + 2(F_2 + E'_{2,1} + E_{2,1} + G_1 + E_{3,1} + E'_{3,1} + F_3) + E'_{3,2} + E'_{3,3}),$$

which is indicated in Figure 3. The zero divisor  $(u_3)_0$  is the singular fiber of type  $\text{III}^*$  (the bold lines) and the polar divisor  $(u_3)_\infty$  is the singular fiber of type  $\text{I}_6^*$  (the thin lines). The curves  $E_{2,2}, E_{2,3}, E_{3,2}$  and  $E_{3,3}$  (the dotted lines) are all the sections.

Eliminating the variable  $t$  from (2.6) and (4.1), we have the following equation

$$(4.3) \quad y_2^2 = u_3^3(y_1^2 - 1)^2 + 1,$$

which has a rational point  $(y_1, y_2) = (1, 1)$  corresponding to the curve  $E_{2,2}$ . Thus, choosing  $E_{2,2}$  as the zero section of the group structure, we obtain the Weierstrass equation for Fibration 3

$$(4.4) \quad Y^2 = X^3 + 4u_3^3X^2 - 4u_3^3X,$$

where the change of variables is given by

$$(4.5) \quad X = \frac{2(y_2 + 1)}{(y_1 - 1)^2}, \quad Y = \frac{4(u_3^3(y_1 + 1)(y_1 - 1)^2 + y_2 + 1)}{(y_1 - 1)^3}.$$

Besides the above two singular fibers of types  $\text{III}^*$  and  $\text{I}_6^*$ , the fibration has three  $\text{I}_1$  fibers at  $u_3 = -1, -\omega$  and  $-\omega^2$ .

The 2-torsion rational point  $(X, Y) = (0, 0)$  corresponds to the curve  $E_{3,3}$ . The rational point  $(X, Y) = (1, -1)$  corresponds to the curve  $E_{3,2}$  of height  $\langle E_{3,2}, E_{3,2} \rangle = \frac{3}{2}$ , which is a generator of the Mordell-Weil lattice of the fibration. The curve  $E_{2,3}$  is another free section corresponding to the rational point  $(1, 1)$  with the relation  $E_{2,3} = -E_{3,2}$  in the Mordell-Weil group.

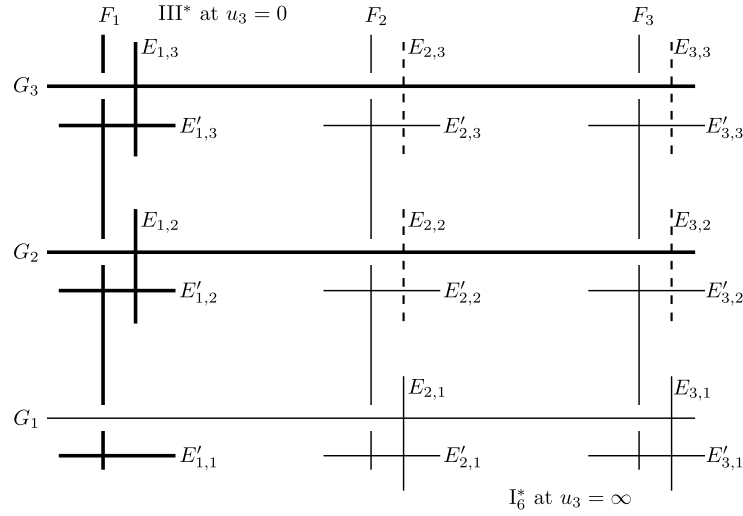


FIGURE 3. Fibration 3

## 5. FIBRATION 5

An elliptic parameter for Fibration 5 is given by

$$(5.1) \quad u_5 = y_1.$$

It is clear that this elliptic parameter defines a fibration having three singular fibers all of types  $\text{IV}^*$  at  $u_5 = 1, -1$  and  $\infty$  (the bold lines in Figure 4) from (2.7). Furthermore the fibration is induced by the composition of the first projection  $C_\omega^1 \times C_\omega^2 \rightarrow C_\omega^1$  and the covering map of degree three  $p_1 : C_\omega^1 \rightarrow \mathbb{P}^1$  in (3.7).

The following simple coordinate change

$$(5.2) \quad X = (u_5^2 - 1)t, \quad Y = (u_5^2 - 1)^2 y_2$$

converts the equation (2.6) into the Weierstrass equation for Fibration 5

$$(5.3) \quad Y^2 = X^3 + (u_5^2 - 1)^4.$$

The curve  $G_1$ ,  $G_2$  and  $G_3$  correspond to the zero section, 3-torsion rational points  $(0, (u_5^2 - 1)^2)$  and  $(0, -(u_5^2 - 1)^2)$ , respectively (the dotted lines in Figure 4).

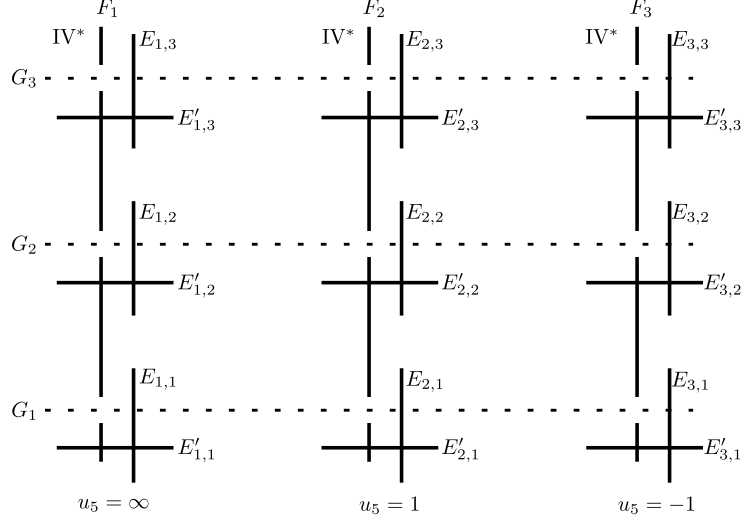


FIGURE 4. Fibration 5

## 6. FIBRATION 6

An elliptic parameter for Fibration 6 is given by

$$(6.1) \quad u_6 = t.$$

Since we gave the divisor of  $t$  in (2.7), we know that the zero divisor  $(u_6)_0$  is the singular fiber of type  $I_{12}$  (the bold lines in Figure 5) and the polar divisor  $(u_6)_\infty$  is the singular fiber of type  $I_3^*$  (the thin lines in Figure 5). The curves  $E_{1,2}, E_{1,3}, E'_{2,1}$  and  $E'_{3,1}$  (the dotted lines in Figure 5) are all the sections. Choosing  $E_{1,2}$  as the zero section of the group structure, we obtain the Weierstrass equation for Fibration 6

$$(6.2) \quad Y^2 = X^3 - 2(u_6^3 - 2)X^2 - u_6^6 X,$$

where the change of variables is given by

$$(6.3) \quad X = \frac{t^3(y_2 + 1)}{y_2 - 1}, \quad Y = \frac{2t^3 y_1(y_2 + 1)}{y_2 - 1}.$$

Besides the two singular fibers of type  $I_{12}$  at  $u_6 = 0$  and of type  $I_3^*$  at  $u_6 = \infty$ , there are three  $I_1$  fibers at  $u_6 = 1, \omega$  and  $\omega^2$ . The Mordell-Weil group of the fibration is isomorphic to  $\mathbb{Z}/4\mathbb{Z}$ . The curve  $E_{1,3}$  corresponds to the rational point  $(0, 0)$  of order two, and remaining curves  $E'_{2,1}$  and  $E'_{3,1}$  correspond to the rational points  $(u_6^3, 2u_6^3), (u_6^3, -2u_6^3)$  of order four, respectively.

## 7. FIBRATION 4

To obtain the Weierstrass equation for Fibration 4, we use a 2-neighbor step from Fibration 3. For more detail about *2-neighbor step*, we refer to [5, 9, 11].

We compute explicitly the elements of  $\mathcal{O}_{X_3}(F)$  where

$$(7.1) \quad \begin{aligned} F = & E_{2,2} + G_2 + E_{1,2} + E'_{1,2} + F_1 + E'_{1,3} + E_{1,3} + G_3 + E_{3,3} + E'_{3,3} + F_3 \\ & + E'_{3,1} + E_{3,1} + G_1 + E_{2,1} + E'_{2,1} + F_2 + E'_{2,2} \end{aligned}$$



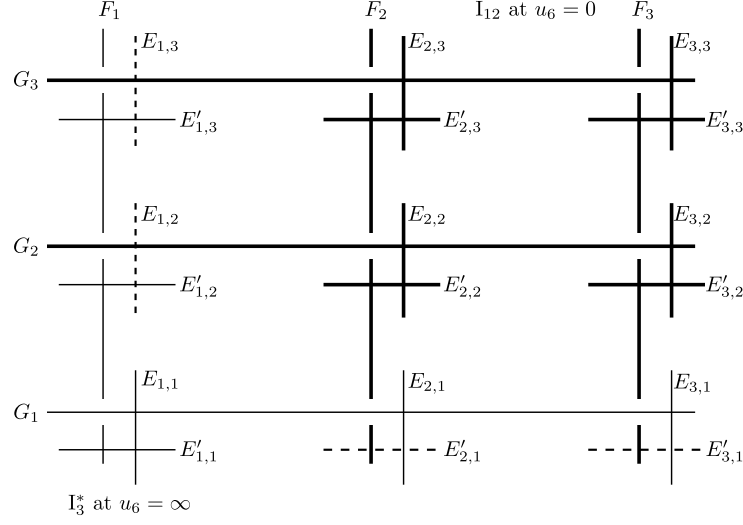


FIGURE 5. Fibration 6

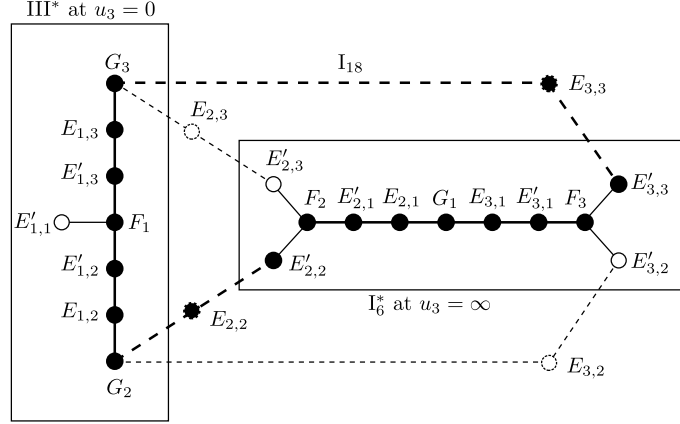


FIGURE 6. 2-neighbor from Fibration 3 to Fibration 4

is the class of the fiber of type  $I_{18}$  we are considering. The linear space  $\mathcal{O}_{X_3}(F)$  is 2-dimensional, and the ratio of two linearly independent elements is an elliptic parameter for  $X_3$ . Since 1 is an element of  $\mathcal{O}_{X_3}$ , we may find a non-constant element of  $\mathcal{O}_{X_3}(F)$ . Then it will be an elliptic parameter of Fibration 4. Let us  $u'_4 \in \mathcal{O}_{X_3}(F)$  be a non-constant. The function  $u'_4$  has a simple pole along  $E_{2,2}$  and  $E_{3,3}$ , which are the zero section and 2-torsion of Fibration 3. Also, it has a simple pole along  $G_2$ , the identity component of the fiber at  $u_3 = 0$ , a simple pole along

$E'_{3,3}$ , the identity component of the fiber at  $u_3 = \infty$ . Therefore we can put

$$(7.2) \quad u'_4 = \frac{\frac{Y}{X} + A_0 + A_1 u_3 + A_2 u_3^2}{u_3},$$

where the variables  $u_3, X, Y$  are given by (4.1) and (4.5). Assume  $A_1 = 0$ , since 1 is an element of  $\mathcal{O}_{X_3}(F)$ . To obtain the coefficients  $A_0$  and  $A_2$ , we look at the order of vanishing along the non-identity components of fibers at  $u_3 = \infty$ . The function  $u'_4$  does not have any pole along  $E'_{3,2}$ , which intersects with the section  $E_{3,2}$  of the fibration 3 at  $u_3 = \infty$ . Hence  $u'_4$  has no pole at  $(X, Y, u_3) = (1, -1, \infty)$ , and that gives us  $A_2 = 0$ . Similarly, the component  $E'_{2,3}$ , which intersects with the section  $E_{2,3}$ , gives us  $A_0 = 0$ . Consequently, we have a new elliptic parameter

$$(7.3) \quad u'_4 = \frac{Y}{u_3 X},$$

where the variables  $u_3, X, Y$  are given by (4.1) and (4.5). Solving for  $Y$  and substituting into the Weierstrass equation (4.4), after suitable coordinate changes we have the following

$$(7.4) \quad y^2 = x^3 + \frac{1}{4}(u'_4{}^2 x - 16)^2.$$

Although this is a Weierstrass equation for Fibration 4, for latter calculations, we put

$$(7.5) \quad u'_4 = \frac{2}{u_4}, \quad x = \frac{2^2 X}{u_4^4}, \quad y = \frac{2^3 Y}{u_4^6}$$

and obtain another Weierstrass equation for Fibration 4

$$(7.6) \quad Y^2 = X^3 + (X - u_4^6)^2.$$

The change of variables is given by

$$(7.7) \quad u_4 = \frac{t}{y_1 + y_2}, \quad X = \frac{(y_1^2 - 1)t^3}{(y_1 + y_2)^4}, \quad Y = \frac{(y_1^2 y_2 + 2y_1 + y_2)t^6}{(y_2^2 - 1)(y_1 + y_2)^6}.$$

The fibration has singular fibers of type  $I_{18}$  at  $u_4 = 0$  and of type  $I_1$  at the zeros of  $27u_4^6 + 4 = 0$ . The zero section corresponds to the divisor  $E'_{1,1}$ . The 3-torsion rational points  $(0, u_4^6)$  and  $(0, -u_4^6)$  correspond to the divisors  $E'_{3,2}$  and  $E'_{2,3}$ , respectively. The free rational points  $(2u_4^3, u_4^4 + 2u_4^3)$  and  $(-2u_4^6, u_4^3 - 2u_4^3)$  correspond to the divisors  $E_{3,2}$  and  $E_{2,3}$ , respectively with the relation  $E_{2,3} + E_{3,2} = E'_{2,3}$  in the Mordell-Weil group. Since the height of  $E_{2,3}$  is equal to  $\frac{3}{2}$ ,  $E_{2,3}$  generates the Mordell-Weil lattice of the fibration.

## 8. FIBRATION 2

We obtain the following elliptic parameter  $u'_2$  for Fibration 2 by a 2-neighbor step from Fibration 4 (see Figure 8).

$$(8.1) \quad u'_2 = \frac{u_4^6 + X + Y}{u_4^2 X}$$

The variables  $u_4, X, Y$  are given by (7.7). Then we get the following Weierstrass equation for Fibration 2.

$$(8.2) \quad y^2 = x^3 + 2(u'_2{}^3 - 4)x^2 + 16x.$$

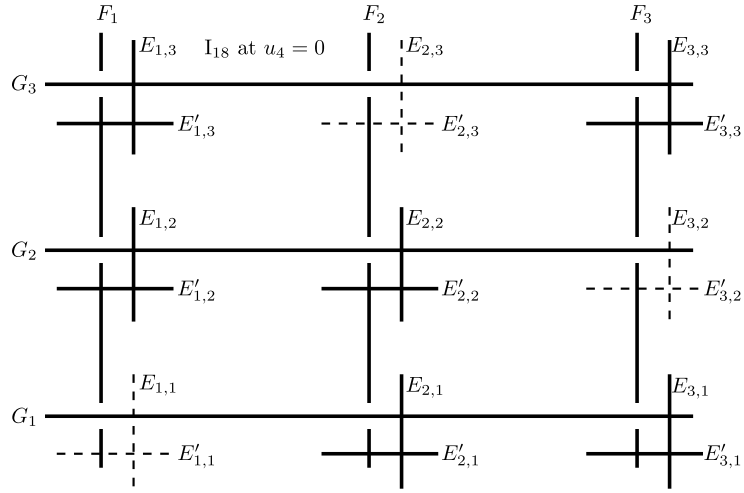


FIGURE 7. Fibration 4

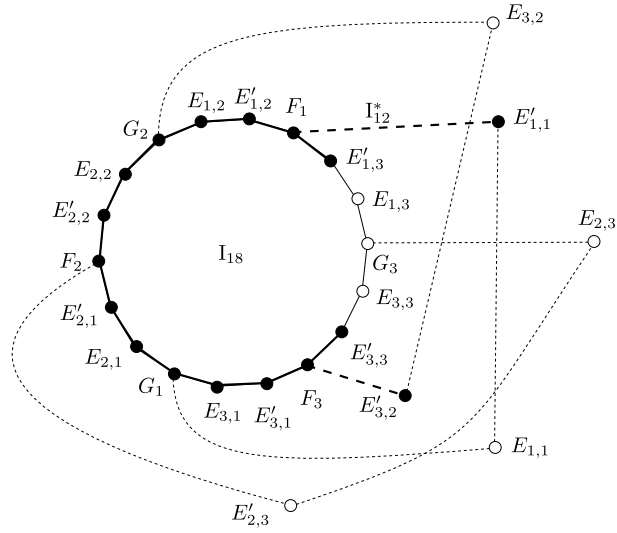


FIGURE 8. 2-neighbor form Fibration 4 to Fibration 2

We put

$$(8.3) \quad u'_2 = \frac{2}{u_2}, \quad x = \frac{2^2 X}{u_2^4}, \quad y = \frac{2^3 Y}{u_2^6}$$

and obtain another Weierstrass equation for Fibration 4.

$$(8.4) \quad Y^2 = X^3 - 2(u_2^3 - 2)X^2 - u_2^8X.$$

The change of variables is given by

$$(8.5) \quad \begin{aligned} u_2 &= \frac{2t^2}{(y_2 + 1)(y_1^2 + 2y_1 + 2y_2 - 1)}, \\ X &= -\frac{32(y_1 - 1)^2(y_2 - 1)^3 t^2}{(y_2 + 1)^2(y_1^2 + 2y_1 + 2y_2 - 1)^4}, \\ Y &= -\frac{128(y_1 - 1)^3(y_2 - 1)^4(y_1 + 1)(y_1 + y_2)}{(y_2 + 1)^2(y_1^2 + 2y_1 + 2y_2 - 1)^5}. \end{aligned}$$

The zero divisor  $(u_4)_0$  is the singular fiber of type  $I_{12}^*$  (the bold lines in Figure 9). The polar divisor  $(u_4)_\infty = G_3 + E_{2,3} + Q_2$  is the singular fiber of type  $I_3$  (the thin lines in Figure 9), where the divisor  $Q_2$  is the lifting of the curve  $y_1^2 + 2y_1 + 2y_2 - 1 = 0$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  by the map  $\pi$  in §3. Besides these two singular fibers, there are three  $I_1$  fibers at  $u_2 = 1, \omega$  and  $\omega^2$ . The zero section corresponds to the divisor  $E_{1,3}$ . The 2-torsion rational point  $(0, 0)$  corresponds to the divisor  $E_{3,3}$ .

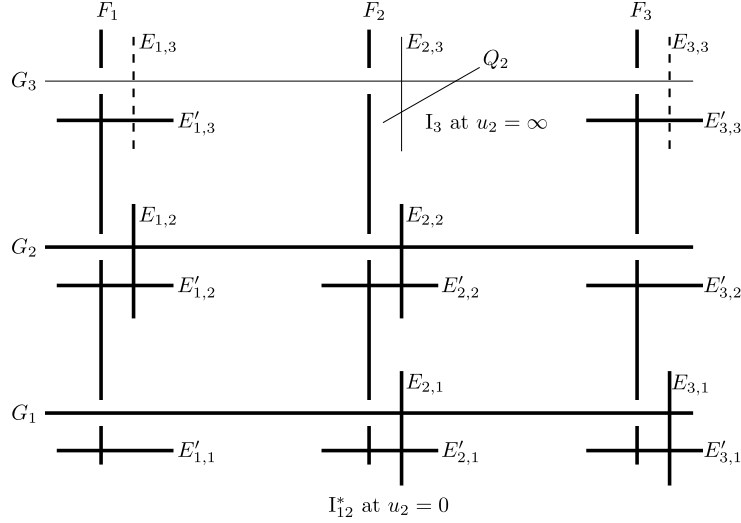


FIGURE 9. Fibration 2

*Remark 2.* We give a Weierstrass equation for Fibration 6 in §6. Comparing the equations (8.4) and (6.2), we know easily that Fibration 2 is a quadratic twist of Fibration 6. This is the reason why we adopt the equation (8.4) as the Weierstrass equation for Fibration 2 rather than the equation (8.2).

**Acknowledgements.** *The computer algebra system Maple and Maple Library “Elliptic Surface Calculator” written by Professor Masato Kuwata [6] were used in the calculation for this paper. The author would like to thank the developers of these programs.*

## REFERENCES

- [1] Sang Yook An, Seog Young Kim, David C. Marshall, Susan H. Marshall, William G. McCallum and Alexander R. Perlis, “Jacobians of genus one curves”, J. Number Theory **90** (2001), no. 2, 304-315.
- [2] A. P. Braun, Y. Kimura and T. Watari, “On the classification of elliptic fibrations modulo isomorphism on  $K3$  surfaces with large Picard number”, arXiv:1312.4421.
- [3] I. Connell, Addendum to a paper of K. Harada and M.-L. Lang, “Some elliptic curves arising from the Leech lattice” [J. Algebra **125** (1989), no. 2, 298310], J. Algebra **145** (1992), 463-467.
- [4] K. Kodaira, “On compact analytic surfaces II”, Ann. of Math. **77**, no.3 (1963), 545-560.
- [5] A. Kumar, “Elliptic fibrations on a generic Jacobian Kummer surface”, arXiv:1105.1715.
- [6] M. Kuwata, “Maple Library ‘Elliptic Surface Calculator’ ”, <http://c-faculty.chuo-u.ac.jp/~kuwata/ESC.php>.
- [7] M. Kuwata and T. Shioda, “Elliptic parameters and defining equations for elliptic fibrations on a Kummer surface”, Algebraic geometry in East Asia-Hanoi (2005), 177-215, Adv. Stud. Pure Math., **50**, Math. Soc. Japan, Tokyo, 2008.
- [8] K. Nishiyama, “The Jacobian fibrations on some  $K3$  surfaces and their Mordell-Weil groups”, Japan. J. Math. (N.S.) **22** (1996), no. 2, 293-347.
- [9] T. Sengupta, “Elliptic fibrations on supersingular  $K3$  surface with Artin invariant 1 in characteristic 3”, arXiv:1204.6478.
- [10] T. Shioda and H. Inose and, On singular  $K3$  surfaces, Complex analysis and algebraic geometry, 119–136. Iwanami Shoten, Tokyo, 1977.
- [11] K. Utsumi, “Weierstrass equations for Jacobian fibrations on a certain  $K3$  surface”, Hiroshima Math. J. **42**, (2012), no. 3, 355-383.

DEPARTMENT OF MATHEMATICS  
 GRADUATE SCHOOL OF SCIENCE  
 HIROSHIMA UNIVERSITY  
 1-3-1 KAGAMIYAMA, HIGASHI-HIROSHIMA 739-8526  
 HIROSHIMA UNIVERSITY  
*E-mail address:* kazu-utsumi@hiroshima-u.ac.jp